Extremal and Probabilistic Graph Theory
Lecture 20

May 12nd, Thursday

We firstly review several theorems proved in previous lectures.

Theorem 20.1. *Let G be a bipartite graph with average degree d and parts U, V with $|U| = |V| = n$. If $r, s, t \geqslant 1$ are such that*

$$n^{-r-s+s^2} d^{-s^2} (t-1)^s < \frac{1}{4},$$

then there exists $X \subseteq U$ and $Y \subseteq V$ of size at least $4^{-\frac{1}{s}} d^s n^{r-s}$ each, satisfying that every r vertices in X or in Y have at least t common neighbors in $G(X, Y)$.

Theorem 20.2. *Let $F = (A, B)$ be bipartite such that any $b \in B$ has degree at most r in F. Then*

$$ex(n, F) \leqslant C \cdot n^{2-\frac{1}{r}}.$$

Definition 20.3. A graph G is r-degenerate, if for any $H \subseteq G$, there exists a vertex v such that $d_H(v) \leqslant r$.

Fact 20.4. *G is r-degenerate if and only if the $(r+1)$-core of G is empty.*

Theorem 20.5. *Let $r \geqslant 2$, and let F be an r-degenerate bipartite graph whose largest part has size t. Then*

$$ex(n, F) \leqslant C \cdot (t-1)^{\frac{1}{2r}} n^{2-\frac{1}{4r}},$$

where C is an absolute constant.

Proof.
Let G be an n-vertex F-free graph with average degree $2d$. Then there exists a bipartite $H = (A, B)$ of G with $|A| = |B| = \frac{n}{2}$ and average degree at least d. Denote $s = 2r$.

If $(A) : \left(\frac{n}{2}\right)^{r-s+s^2} d^{-s^2} (t-1)^s < \frac{1}{4}$ and $(B) : 4^{-\frac{1}{s}} d^s n^{1-s} > t - 1$ hold, then by Theorem 20.1, there exists $X \subseteq A$ and $Y \subseteq B$ such that X and Y are of size at least $4^{-\frac{1}{s}} d^s n^{1-s} \geqslant t$.

Moreover, any r vertices in X or in Y have at least t common neighbors in $H(X, Y)$.

By the previous embedding lemma, we can find a copy of F in (X, Y), a contradiction!(Here we use an extended version of embedding lemma, noticing that F is r-degenerate if and only if $V(F)$ can be listed as $\{v_1, v_2, \cdots, v_n\}$, satisfying that $|N(v_i) \cap \{v_1, v_2, \cdots, v_{i-1}\}| \leqslant r$.)

So (A) or (B) doesn't hold. Since (A) implies (B), we have

$$n^{-r} \left(\frac{n}{d}\right)^{s^2} (t-1)^s > \frac{1}{4}.$$

Thus

$$e(H) = d \cdot n \leqslant (t-1)^{\frac{1}{2r}} n^{2-\frac{1}{4r}}.$$

∎

Corollary 20.6. *For bipartite F, let*

$$d_F = \max_{H \subseteq F} \frac{2e(H)}{v(H)},$$

then

$$ex(n, F) \leqslant O(n^{2-\frac{1}{4\lfloor d_F \rfloor}}) \leqslant O(n^{2-\frac{1}{4d_F}}).$$

Proof. Because such F is $\lfloor d_F \rfloor$-degenerate. ∎

Theorem 20.7 (Erdös-Rényi first moment method; A general lower bound for $ex(n, F)$)**.** *For bipartite F, define*

$$c_F = \min_{H \subseteq F} \frac{v(H)}{e(H)}$$

and

$$r_F = \min_{H \subseteq F} \frac{v(H) - 2}{e(H) - 1},$$

where the minimums are taken only for those $H \subseteq F$ with $e(H) \geqslant 1$ and $v(H) \geqslant 2$ respectively.

Then

$$ex(n, F) \geqslant \Omega(n^{2-r_F}) \geqslant (n^{2-c_F}).$$

Proof.

We can easily argue that $c_F \geqslant r_F$.

Let $H \subseteq F$ be the subgraph of F attaining the minimum of r_F.

Let $v = v(H)$ and $e = e(H)$. Let α be the number of copies of H in K_v.

Consider a random graph $G(n, p)$.

Let β denote the number of copies of H in $G(n, p)$.

So

$$E(\beta) = \sum_{K \in \binom{[n]}{v}} E[\text{number of copies } H \text{ in } K] = \binom{n}{v} \cdot \alpha \cdot p^e,$$

and $E[e(G)] = p\binom{n}{2}$.

Taking $p = c' \cdot n^{-r_F}$ such that

$$E[e(G)] - E[\beta] = \binom{n}{2} p - \binom{n}{2} \alpha p^e \geqslant \frac{1}{2} \binom{n}{2} p,$$

i.e.

$$\frac{1}{2} \binom{n}{2} p \geqslant \binom{n}{v} \alpha p^e,$$

which is equivalent to

$$n^{r_F(e-1)} \geqslant n^{v-2}.$$

So there exists a graph G such that

$$e(G) - \beta \geqslant \frac{1}{2}\binom{n}{2}p = \Omega(n^{2-r_F}).$$

Let G' be the graph obtained from G by deleting an edge from each copy of H.
Then G' is H-free and $e(G') \geqslant e(G) - \beta \geqslant \Omega(n^{2-r_F}) \geqslant \Omega(n^{2-c_f})$.
But $H \subseteq F$, so G' is also F-free.

∎

Theorem 20.8. *For any bipartite F,*

$$\Omega(n^{2-c_F}) \leqslant ex(n,F) \leqslant O(n^{2-\frac{c_F}{8}}),$$

where

$$c_F = \min_{H \subseteq F} \frac{v(H)}{e(H)}.$$

Proof. Because $\frac{d_F}{2} = \frac{1}{c_F}$, this follows from Corollary 20.6 and Theorem 20.7. ∎

Remark. This lower bound is still best for general F.

Theorem 20.9. *Let $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$, where F_j is bipartite. Let*

$$c = \max_j c_{F_j} = \max_j \min_{H \subseteq F_j} \frac{v(H)}{e(H)},$$

and

$$r = \max_j r_{F_j} = \max_j \min_{H \subseteq F_j} \frac{v(H)-2}{e(H)-1}.$$

Then

$$ex(n,F) \geqslant \Omega(n^{2-r}) \geqslant \Omega(n^{2-c}).$$

Proof. Left as an exercise. ∎

Remark. Theorem 20.9 implies that

$$ex(n, \{C_3, C_4, \cdots, C_m\}) \geqslant \Omega(n^{1+\frac{1}{m-1}});$$

and thus

$$\Omega(n^{1+\frac{1}{2t-1}}) \leqslant ex(n, \{C_3, C_4, \cdots, C_m\}) \leqslant ex(n, C_{2t}) \leqslant O(n^{1+\frac{1}{t}}).$$

The best lower bound for C_{2t} is given by the following theorem.

Theorem 20.10 (Lazebnik-Ustimenko-Woldar)**.** *For $t \geqslant 2$,*

$$ex(n, C_{2t}) \geqslant ex(n, \{C_3, C_4, \cdots, C_{2t+1}\}) \geqslant \begin{cases} \Omega(n^{1+\frac{2}{3t-3}}) & \text{if } t \text{ is odd;} \\ \Omega(n^{1+\frac{2}{3t-2}}) & \text{if } t \text{ is even.} \end{cases}$$

And when $t = 2, 3$ or 5,

$$ex(n, C_{2t}) = \Theta(n^{1+\frac{1}{t}}).$$

Remark. It is one of the main problems in extremal combinatorics to determine $ex(n, C_{2t})$ for $t \notin \{2, 3, 5\}$.

Definition 20.11. For graph F, the Ramsey number $r(F)$ is the minimum n such that any 2-edge-coloring of K_n has a monochromatic copy of F.

Theorem 20.12. *Let $F = (A, B)$ ba a bipartite graph such that any $b \in B$ has degree at most r. Then*

$$r(F) \leqslant 4^r(n + r).$$

Proof.

Consider any 2-edge-coloring of K_N. Let G_i be the graph defined by edges of color $i \in \{1, 2\}$. Since $E(K_N) = E(G_1) \cup E(G_2)$, we can assume $e(G_1) \geqslant \frac{1}{2}\binom{N}{2}$.

If G_1 is F-free, then by Theorem 20.2,

$$e(G_1) \leqslant ex(N, F) \leqslant (n + r)^{\frac{1}{r}} N^{2-\frac{1}{r}}.$$

It follows that

$$\frac{1}{2}\binom{N}{2} \leqslant e(G_1) \leqslant (n + r)^{\frac{1}{r}} N^{2-\frac{1}{r}}.$$

Therefore, $N \leqslant 4^r(n + r)$.

∎

Theorem 20.13. *Let F be an r-degenerate bipartite graph with n vertices. Then*

$$r(F) \leqslant 2^{2\sqrt{r \log n}} n.$$

Proof. Left as an exercise. Hint: Using Theorem 20.8. ∎

Remark. Recently, Lee proved that for such F, $r(F) \leqslant c(r) \cdot n$, proving a conjecture of Burr-Erdös.